On the Positone Problem for Elliptic Systems

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1. Introduction. In this article we study the existence and dependence on the parameter λ of solutions to the system

$$(1.1)\qquad \begin{aligned} L^\mu u^\mu &= \lambda f^\mu(x,u) && \text{in } \Omega \\ u^\mu &= 0 && \text{on } \partial\Omega \\ u^\mu &> 0 && \text{in } \Omega,\ \mu = 1, \ldots, k, \end{aligned}$$

where $u = (u^1,\ldots,u^k)$, $\Omega \subseteq \mathbf{R}^n$ is a smooth, bounded domain, and for each μ,

$$L^\mu = -\sum_{i,j=1}^{n} a_{ij}^\mu(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^\mu(x)\frac{\partial}{\partial x_i} + c^\mu(x)$$

is a uniformly elliptic differential operator with Hölder continuous coefficients, and $c^\mu \geq 0$. Our work has two objectives: first, to extend the results of Cohen and Keller [4] (see also [8]) on the single equation

$$(1.2)\qquad \begin{aligned} Lu &= \lambda f(x,u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \\ u &> 0 && \text{on } \Omega \end{aligned}$$

to systems of the form (1.1); and second, to present a more modern viewpoint on the problem and to exploit results and methods developed since the publication of [4].

In [4], Keller and Cohen studied (1.2) under the assumptions $f(x,0) > 0$ and $f(x,u)$ increasing in u. They showed that (1.2) has a solution for $\lambda \in (0,\lambda^*)$, some $\lambda^* > 0$, and constructed a minimal solution by monotone iteration. Under additional conditions on f they obtained estimates on λ^*, conditions under which a solution exists for $\lambda = \lambda^*$, and uniqueness results. Specifically, if $f(x,u)$ is concave in u, no solution exists for $\lambda = \lambda^*$ and the solution for $\lambda \in (0,\lambda^*)$ is unique. The convex case is more complicated; in some cases solutions are not

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unique. Results on the convex case have been obtained by various workers; see [1], [5], [7], and for a survey of results and additional references, [10]. In the present work, we restrict our attention to results of the type obtained in [4]; we plan to make further investigations of the convex case.

Our approach to (1.1) follows that taken in [4] for (1.2). The main tools are the theory of eigenvalue problems for linear operators derived from (1.1) and monotonicity or comparison arguments. In Section 2, we present results on the solvability and properties of solutions for

$$(1.3)\qquad \begin{aligned} L^{\mu}u^{\mu} &= \lambda \sum_{\nu=1}^{k} m^{\mu\nu}(x)u^{\nu} + h^{\mu}(x) \qquad \text{in } \Omega, \\ u^{\mu} &= 0 \qquad \text{on } \partial\Omega,\ \mu = 1, \ldots, k. \end{aligned}$$

The recent work of Hess [6] on eigenvalue problems with indefinite weight functions allows us to obtain the results we need to study (1.1) under conditions weaker than requiring that $f^{\mu}(x,0) > 0$ for all μ, all $x \in \Omega$, thereby extending the results of [4] even in the case of a single equation. In Section 3, we show that under suitable hypotheses, (1.1) has a minimal solution for $\lambda \in (0,\lambda^*)$. A question that arises here is the definition of positivity. We look for solutions positive in all components and impose conditions on $f^{\mu}(x,u)$ so that such solutions exist. This type of question for linear systems such as (1.3) was investigated in [3]; some of the results of [3] are used in Section 2. In Section 4, we obtain estimates on λ^*, show that the solution to (1.1) is unique for concave nonlinearities, and that for concave nonlinearities there is no solution for $\lambda = \lambda^*$. In working with systems, the conditions we impose on the nonlinearity become more complicated than those for a single equation. However, our hypotheses reduce to those of [4] or weaker ones when specialized to a single equation.

Remark on notation. In what follows, we will not explicitly distinguish vectors from scalars by notation. Which type of quantity a variable represents will either be stated or will follow readily from the context. In our analysis we will make extensive use of the notion of ordering. Our notation can best be described in terms of ordered Banach spaces. Let E be an ordered Banach space with positive cone P; that is, let E be a real Banach space and let $P \subseteq E$ be such that $\mathbf{R}^+P \subseteq P$, $P + P \subseteq P$, and $P \cap (-P) = \{0\}$. If u, $v \in E$, we write $u \geq v$ if $u - v \in P$, and $u > v$ if $u - v \in P \setminus \{0\}$. For Euclidean spaces $\mathbf{R}^m$ we take $P = (\bar{\mathbf{R}}^+)^m$; for function spaces (typically Sobolev or Hölder spaces) we take $P = \{u \in E : u(x) \geq 0 \text{ on } \Omega\}$. We will also use a more specialized notation: if u, $v \in C^1(\bar{\Omega})$, we will write $u >> v$ if $u(x) \geq v(x)$ on $\bar{\Omega}$, $u(x) > v(x)$ on Ω, and $\partial u/\partial n < \partial v/\partial n$ on $\partial\Omega$. If u, $v \in [\mathbf{C}^1(\bar{\Omega})]^k$ we write $u >> v$ if $u^{\mu} >> v^{\mu}$ for all components of u and v. (Note that $u >> 0$ does not imply $u \in \text{int } P$, since we may have $u = 0$ on $\partial\Omega$. However, if $u >> 0$ and $v \in [C^1(\bar{\Omega})]^k$ with $v = 0$ on $\partial\Omega$, then $u \pm v >> 0$ if $\|v\|$ is sufficiently small. Since we will be dealing with Dirichlet boundary conditions, the notation ">>" will be quite useful).

2. The Positivity Lemma. In this section we establish a lemma concerning the solvability of certain nonhomogeneous linear elliptic systems. To this end, for $\mu = 1, \ldots, k$, let L^μ be the differential operator

$$L^\mu = -\sum_{i,j=1}^{n} a_{ij}^\mu(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^\mu(x) \frac{\partial}{\partial x_i} + c^\mu(x), \tag{2.1}$$

where $x \in \Omega$, a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. Assume that the functions a_{ij}^μ, b_i^μ, c^μ are of class $C^\alpha(\bar{\Omega})$, that

$$\sum_{i,j=1}^{n} a_{ij}^\mu(x)\eta_i\eta_j \geq K^\mu|\eta|^2,$$

where $\eta = (\eta_i, \ldots, \eta_n) \in \mathbf{R}^n$ and $K^\mu > 0$, and that $c^\mu(x) \geq 0$ for all $x \in \bar{\Omega}$.

Let $M(x)$ be a $k \times k$ matrix with $m^{\mu\nu}$ of class $C^\alpha(\bar{\Omega})$ and $m^{\mu\nu}(x) \geq 0$ for all $x \in \bar{\Omega}$. Let $h: \bar{\Omega} \to (\bar{\mathbf{R}}^+)^k$ be of class $C^\alpha(\bar{\Omega})$.

We now consider the system

$$\begin{aligned} Lu &= \lambda M u + h, \qquad x \in \Omega \\ u &= 0, \qquad x \in \partial\Omega, \end{aligned} \tag{2.2}$$

where

$$L = \begin{pmatrix} L^1 & & 0 \\ & \cdot & \\ & \cdot & \\ 0 & \cdot & L^k \end{pmatrix}, \qquad u = \begin{pmatrix} u^1 \\ \cdot \\ \cdot \\ \cdot \\ u^k \end{pmatrix},$$

and for each μ, $u^\mu \in C^{2+\alpha}(\bar{\Omega})$. Our result is

Lemma 2.1. *(i) Suppose there is $x_0 \in \Omega$ and $\mu \in \{1, \ldots, k\}$ such that $m^{\mu\mu}(x_0) > 0$. Then the system*

$$\begin{aligned} Lv &= \lambda M v, \qquad x \in \Omega \\ v &\equiv 0, \qquad x \in \partial\Omega \end{aligned} \tag{2.3}$$

has a smallest positive eigenvalue λ_0 admitting a nonnegative solution v. Furthermore, if $\lambda < \lambda_0$, (2.2) has a unique nonnegative solution for any $h \geq 0$. If $\lambda \geq \lambda_0$, (2.2) has no nonnegative solution provided $h^\mu(x_\mu) > 0$ for some $x_\mu \in \Omega$, $\mu = 1, \ldots, k$.

(ii) If, in addition, there is $x_0 \in \Omega$ such that $M(x_0)$ is irreducible and $m^{\mu\mu}(x_0) > 0$ for $\mu = 1, \ldots, k$, (2.2) has a nonnegative solution for $h > 0$ only in the case $\lambda < \lambda_0$. In this case, the solution u is such that $u >> 0$.

Proof. (i) That (2.3) has a smallest positive eigenvalue admitting a nonnegative solution is the content of [6] and [3]. Theorem 2.16 of [9] may then be employed to assert existence of a unique nonnegative solution in case $\lambda < \lambda_0$.

Suppose now we have a nonnegative solution in case $\lambda \geq \lambda_0$ and h is as hypothesized. Consider the equivalent system

$$\frac{1}{\lambda} u = L^{-1} M u + \frac{1}{\lambda} L^{-1} h. \tag{2.4}$$

The spectral radius of the operator $L^{-1}M$ is $1/\lambda_0$.

The Krein-Rutman theorem guarantees the existence of a positive linear functional f^* such that $(L^{-1}M)^* f^* = (1/\lambda_0) f^*$, where $(L^{-1}M)^*$ is the dual of $L^{-1}M$. Then

$$\frac{1}{\lambda} f^*(u) = \frac{1}{\lambda_0} f^*(u) + \frac{1}{\lambda} f^*(L^{-1}h).$$

Since $u \geq 0$, $f^*(u) \geq 0$. Hence $(1/\lambda - 1/\lambda_0) f^*(u) \leq 0$.

However, the hypothesis on h guarantees that $f^*(L^{-1}h) > 0$, a contradition, which establishes (i).

(ii) Note that $M^{k-1}(x_0)$ is a matrix which is positive in all its entries, by [2, p. 27]. Observe then that if $h > 0$, $L^{-1}h > 0$ and $(L^{-1}h)^{\mu_1} >> 0$ for some $\mu_1 \in \{1,\ldots,k\}$ by the strong maximum principle. Then $[M(L^{-1}h)]^{\mu}(x_0) > 0$ for $\mu = \mu_1$ since $m^{\mu_1\mu_1}(x_0) > 0$ and also for some $\mu_2 \in \{1,2,\ldots,k\}$, $\mu_2 \neq \mu_1$, since $M(x_0)$ is irreducible. The strong maximum principle implies that L^{-1} preserves componentwise positivity. In fact, $[L^{-1}M(L^{-1}h)]^{\mu} >> 0$ for $\mu = \mu_1, \mu_2$. We now repeat the argument, applying M once more. We obtain that $[M(L^{-1}M)(L^{-1}h)]^{\mu}(x_0) > 0$ for $\mu = \mu_1, \mu_2$, and μ_3, $\mu_3 \neq \mu_1, \mu_2$. A repeated application of the strong maximum principle guarantees that $[(L^{-1}M)^2(L^{-1}h)]^{\mu} >> 0$ for $\mu = \mu_1, \mu_2$ and μ_3. After $(k - 1)$ iterations, we can guarantee that $(L^{-1}M)^{k-1}(L^{-1}h) >> 0$. Hence $(L^{-1}M)^{k-1}(L^{-1}h) > \varepsilon v$, for some $\varepsilon > 0$, where $v >> 0$ satisfies (2.3) with $\lambda = \lambda_0$. By considering (2.4) and applying Theorem 2.16 of [9], we see that (2.2) has solutions for $h > 0$ if and only if $\lambda < \lambda_0$. That such solutions are componentwise positive follows as in the proof of Lemma 3.1 of [3].

Remark 2.2. Lemma 2.1 is the system analogue to the Positivity Lemma of [4]. Our proof shows that the hypothesis in [4] may be reduced to the assumption that the coefficient function p is positive for some $x_0 \in \Omega$.

Corollary 2.3. *Suppose that M has the form*

$$M = \begin{pmatrix} M^1 & 0 & \cdots & 0 \\ 0 & M^2 & & \vdots \\ & & \ddots & \\ \vdots & & & \vdots \\ & & \ddots & \\ \vdots & & & \vdots \\ 0 & \cdots & & M^N \end{pmatrix},$$

where M^{γ} is an $r_\gamma \times r_\gamma$ matrix satisfying the hypothesis of Lemma 2.1(ii), with $\sum_{\gamma=1}^{N} r_\gamma = k$. Let $\lambda_0\,(M^{\gamma})$ denote the smallest eigenvalue of the system

$$L^\gamma w = \lambda M^\gamma w, \tag{2.5}$$

where

$$L^\gamma = \begin{pmatrix} L^{\delta\gamma+1} & & & 0 \\ & \cdot & & \\ 0 & & \cdot & \\ & & & L^{\delta\gamma+r\gamma} \end{pmatrix}$$

and

$$\delta\gamma = \begin{cases} 0, & \text{if } \lambda = 1 \\ \sum_{j=1}^{\gamma-1} r_j, & \text{if } 2 \le \gamma \le N. \end{cases}$$

Then if $h^{\delta\gamma+m} \neq 0$ *for some* $m \in \{1,\ldots,r_\gamma\}$, *(2.2) has a nontrivial solution* w *with*

$$\begin{pmatrix} u^{\delta\gamma+1} \\ \cdot \\ \cdot \\ \cdot \\ u^{\delta\gamma+r\gamma} \end{pmatrix} \neq 0$$

only if $\lambda < \lambda_0(M^\gamma)$. *Furthermore, in this case,* $u^{\delta\gamma+m}(x) > 0$ *for* $x \in \Omega$, *and* $m = 1, \ldots, r_\gamma$.

3. Main results. In this section we discuss the basic existence theory for positive solutions of the system

$$\begin{aligned} Lu &= \lambda f(x,u) \qquad x \in \Omega \\ u &= 0 \qquad x \in \partial\Omega, \end{aligned} \tag{3.1}$$

where L and Ω are as in Section 2, and $f = (f^1,\ldots,f^k)^T$. We take the approach used by Cohen and Keller [4] for a single equation, namely monotone iteration. We assume that $f(x,u)$ is of class C^α in both x and u and satisfies the following hypotheses:

F1 $$f(x,\cdot): (\overline{\mathbf{R}^+})^k \to (\overline{\mathbf{R}^+})^k \text{ for all } x \in \bar{\Omega}.$$

F2 For each $\mu \in \{1,\ldots,k\}$ there is a sequence $\nu_0, \nu_1, \ldots \nu_N$ in $\{1,\ldots,k\}$ with $\nu_N = \mu$ such that $f^{\nu_0}(x,0) > 0$ for some $x \in \Omega$, and if $u: \bar{\Omega} \to (\overline{\mathbf{R}^+})^k$ with $u^{\nu_j}(x) > 0$ on Ω for $j = 0, \ldots, J$, $J \le N - 1$, then $f^{\nu_{J+1}}(x,u) > 0$ for some $x \in \Omega$.

F3 $$f^\mu(x,u) \text{ is nondecreasing in } u^\nu \text{ for } \nu \neq \mu.$$

F4 There is a constant $K \ge 0$ such that for all μ, if $u^\nu = \bar{u}^\nu \ge 0$ for $\nu \neq \mu$ and $u^\mu \ge \bar{u}^\mu$, then

$$f^\mu(x,u) - f^\mu(x,\bar{u}) \ge -K(u^\mu - \bar{u}^\mu).$$

Conditions F3 and F4 are the monotonicity hypotheses needed to obtain an increasing sequence of iterates. Condition F2 is a nonlinear analogue of the irreducibility condition required for matrices in Section 2; it guarantees the positivity of all components of the solution, but is weaker than requiring $f^{\mu}(x,0) > 0$ for all μ.

Example. The system

$$-\Delta u^1 = \lambda(1 + u^1)$$
$$-\Delta u^2 = \lambda(u^1 + u^2)$$

satisfies F1–F4, even though $f^2(x,0) \equiv 0$. For the system

$$-\Delta u^1\ \lambda(1 + u^1)$$
$$-\Delta u^2 = \lambda u^2,$$

condition F2 fails and the system decouples into two single equations, the second of which has only the zero solution for λ small.

A slightly more general problem similar to (3.1) is

$$L^{\mu}u^{\mu} = \lambda^{\mu} f^{\mu}(x,u) \qquad \text{in } \Omega,\ \mu = 1, \ldots, k$$
$$u = 0 \qquad \text{on } \partial\Omega. \tag{3.2}$$

We can obtain some information about (3.2) by studying (3.1); specifically, if the k-tuple of parameters $(\lambda^1,\ldots,\lambda^k)$ varies along a ray, we may choose a unit vector e in the direction of the ray, and write $\lambda^{\mu} = \lambda e^{\mu}$. Then (3.2) corresponds to (3.1) with f^{μ} replaced by $e^{\mu}f^{\mu}$.

We will study the set Λ of $\lambda \in \mathbf{R}$ such that (3.1) has a solution positive in all components. It is clear by the maximum principle that when F1 holds, $\lambda > 0$ for all $\lambda \in \Lambda$. We will follow [4] and let $\lambda^* = \sup \Lambda$. In what follows we will see that either $\Lambda = (0,\lambda^*]$, or $\Lambda = (0,\lambda^*)$, with the inclusion of λ^* in Λ depending on the convexity or concavity of $f(x,u)$.

A criterion for existence of solutions to (3.1) which are positive in all components is given by the following result, which is an extension to systems of Theorem (3.2) of [4].

Theorem 3.1. *Suppose that $f(x,u)$ satisfies F1–F4. The iteration scheme*

$$u_0 = 0,$$
$$Lu_{n+1} + \lambda K u_{n+1} = \lambda(f(x,u_n) + K u_n) \qquad \text{in } \Omega \tag{3.3}$$
$$u_{n+1} = 0 \qquad \text{on } \partial\Omega$$

produces a sequence which is increasing in each component. A number $\lambda > 0$ belongs to Λ if and only if the sequence $\{u_n\}$ is uniformly bounded; in that case, the sequence converges uniformly to a solution $\underline{u}(\lambda,x)$ of (3.1). The solution $\underline{u}(\lambda,x)$ is minimal in the sense that $\underline{u}(\lambda,x) \le u(x)$ for any other positive solution u.

Proof. The maximum principle insures that $u_n^\mu \geq 0$ for all n, μ since f satisfies F1. Thus $u_1^\mu \geq u_0^\mu = 0$ for all μ. Suppose $u_n^\mu \geq u_{n-1}^\mu$ for all μ. Fix μ and let $\tilde{u}_n^\nu = u_n^\nu$ for $\nu \neq \mu$, $\tilde{u}_n^\mu = u_{n-1}^\mu$. Then we have

$$(3.4) \quad (L^\mu + \lambda K)(u_{n+1}^\mu - u_n^\mu)$$
$$= \lambda[f^\mu(x,u_n) - f^\mu(x,u_{n-1}) + K(u_n^\mu - u_{n-1}^\mu)]$$
$$= \lambda[f^\mu(x,u_n) - f^\mu(x,\tilde{u}_n) + f^\mu(x,\tilde{u}_n) - f^\mu(x,u_{n-1}) + K(u_n^\mu - u_{n-1}^\mu)].$$

But $f^\mu(x,u_n) - f^\mu(x,\tilde{u}_n) \geq -K(u_n^\mu - u_{n-1}^\mu)$ by F4, $f^\mu(x,\tilde{u}_n) - f^\mu(x,u_{n-1}) \geq 0$ by F3, so $(L^\mu + \lambda K)(u_{n-1}^\mu - u_n^\mu) \geq 0$ in Ω. Since $u_{n+1}^\mu - u_n^\mu = 0$ on $\partial\Omega$, the maximum principle implies that $u_{n+1}^\mu \geq u_n^\mu$. Hence the sequence increases, by induction. If μ is such that F2 holds with $\nu_0 = \mu$, that is, $f^\mu(x,0) > 0$ for some $x \in \Omega$, then since $(L^\mu + \lambda K)u_1^\mu = \lambda f^\mu(x,0)$ in Ω and $u_1^\mu = 0$ on $\partial\Omega$, the strong maximum principle implies that $u_1^\mu(x) > 0$ in Ω. Since the sequence increases, $u_n^\mu(x) > 0$ in Ω for all n. By F2, at least one μ must have $f^\mu(x,0) > 0$ for some x. Thus u_1 is positive in at least one component; also, if $u_1^{\nu_0}(x) > 0$ in Ω then $f^{\nu_1}(x,u_1) > 0$ for some $x \in \Omega$ so $u_2^{\nu_1}(x) > 0$ in Ω by the strong maximum principle, and so on, so that after finitely many iterations all components of u_n are positive in Ω.

Assume now that the sequence $\{u_n\}$ is uniformly bounded above componentwise by v. Then for each x, there is a $u(x) \leq v(x)$ such that $u_n(x) \uparrow u(x)$. Since $u_n \to u$ pointwise with u bounded, $\lambda(f(x,u_n) + Ku_n) \to \lambda(f(x,u) + Ku)$ pointwise and hence in L^p for any $p \in (1,\infty)$. Since the operators $L^\mu + \lambda K$ with Dirichlet boundary conditions can be inverted on L^p, and $(L^\mu + \lambda K)^{-1}: L^p \to W^{2,p}$, we have that $u_n \to u$ in $W^{2,p}$, and hence (by the embedding properties of $W^{2,p}$) in $C^{1+\alpha}$. But then

$$\lambda(f(x,u_n) + Ku_n) \to \lambda(f(x,u) + Ku)$$

in C^α; so since $(L^\mu + \lambda K)^{-1}: C^\alpha \to C^{2+\alpha}$, $u_n \to u$ in $C^{2+\alpha}$ and it follows by passing to the limit in (3.3) that u is a solution of (3.1), which we denote by $\underline{u}(\lambda,x)$.

If w is a solution to (3.1) which is positive in each component, then

$$w^\mu > u_0^\mu = 0,$$

and if $w^\mu \geq u_n^\mu$, then

$$(L^\mu + \lambda K)(w^\mu - u_{n+1}^\mu) = \lambda[f^\mu(x,w) - f^\mu(x,u_n) + K(w^\mu + u_n^\mu)]$$

so arguing as in (3.4) and the discussion following, we have $w^\mu \geq u_{n+1}^\mu$. Thus by induction, w is an upper bound for the sequence u_n and since $w \geq u_n$, $w^\mu(x) \geq \underline{u}^\mu(\lambda,x)$ if $\underline{u}(\lambda,x) = \lim_{n\to\infty} u_n(x)$.

Most of the remaining results of this section are based on the idea of comparing solutions of (3.1) for different choices of f or different values of λ. The following lemma is crucial:

Lemma 3.2. *Suppose that $f(x,u)$ satisfies F1–F4 and $F(x,v)$ is such that whenever $v \geq u \geq 0$*

$$F(x,v) + Kv \geq f(x,u) + Ku.$$

If $v(\lambda,x)$ *is a solution of*

$$\text{(3.5)} \qquad \begin{aligned} Lv &= \lambda F(x,v) && \text{in } \Omega \\ v &= 0 && \text{on } \partial\Omega \end{aligned}$$

with $v \geq 0$, *then* $\lambda \in \Lambda$ *for (3.1) and* $\underline{u}(\lambda,x) \leq v(x)$.

Proof. We compare v with the iterates u_n constructed in proving Theorem 3.1. First, $v \geq u_0 = 0$. Then if $v \geq u_n$, we have

$$(L^\mu + \lambda K)(v^\mu - u^\mu_{n+1}) = \lambda[F^\mu(x,v) - f^\mu(x,u_n) + K(v^\mu - u^\mu_n)] \geq 0$$

in Ω, with $v - u_n = 0$ on $\partial\Omega$, so that $v^\mu \geq u^\mu_{n+1}$. Hence, by induction, v provides an upper bound for $\{u_n\}$ so by Theorem 3.1, $\lambda \in \Lambda$ with $v(x) \geq \underline{u}(\lambda,x)$.

We can now show that Λ is an interval and $u(\lambda,x)$ is nondecreasing componentwise in λ.

Theorem 3.3. *Suppose that* $f(x,u)$ *satisfies F1–F4 and that* $\lambda' \in \Lambda$. *Then* $(0,\lambda') \subseteq \Lambda$. *Further,* $\underline{u}^\mu(\lambda,x)$ *is strictly increasing in* λ *for all* μ.

Proof. Let $F(x,u) = (\lambda'/\lambda)f(x,u)$ where $0 < \lambda < \lambda'$. Then if $v \geq u$ and we let $\bar{u}^\nu = u^\nu$ for $\nu \neq \mu$, $\bar{u}^\mu = v^\mu$, we have

$$\begin{aligned} &(\lambda'/\lambda)f^\mu(x,v) + Kv^\mu - [f^\mu(x,u) + Ku^\mu] \\ &\qquad \geq (\lambda'/\lambda)(f^\mu(x,v) - f^\mu(x,\bar{u})) + f^\mu(x,\bar{u}) - f^\mu(x,u) + K(v^\mu - u^\mu) \\ &\qquad \geq 0 \end{aligned}$$

so that $F(x,v)$ satisfies the hypotheses of Lemma 3.2, and hence $\lambda \in \Lambda$ with $\underline{u}(\lambda',x) \geq \underline{u}(\lambda,x)$.

Now let $w^\mu = \underline{u}^\mu(\lambda',x)$ and $w^\nu = \underline{u}^\nu(\lambda,x)$ for $\nu \neq \mu$. We have $w^\nu > 0$ in Ω for all ν, and also, by F3 and F4,

$$\begin{aligned} &(L^\mu + \lambda K)(\underline{u}^\mu(\lambda',x) - \underline{u}^\mu(\lambda,x)) \\ &\qquad = \lambda' f^\mu(x,\underline{u}(\lambda',x)) - \lambda f^\mu(x,\underline{u}(\lambda,x)) \\ &\qquad\quad + \lambda K(\underline{u}^\mu(\lambda',x) - \underline{u}^\mu(\lambda,x)) \\ &\qquad = \lambda' f^\mu(x,\underline{u}(\lambda',x)) - \lambda' f^\mu(x,w) \\ &\qquad\quad + (\lambda' - \lambda)f^\mu(x,w) + \lambda f^\mu(x,w) - \lambda f^\mu(x,\underline{u}(\lambda,x)) \\ &\qquad\quad + \lambda K(\underline{u}^\mu(\lambda',x) - \underline{u}^\mu(\lambda,x)) \\ &\qquad \geq (\lambda' - \lambda)f^\mu(x,w). \end{aligned}$$

But by F2 and the positivity of w, it follows that $f^\mu(x,w) > 0$ for some $x \in \Omega$. Hence, $\underline{u}^\mu(\lambda',x) > \underline{u}^\mu(\lambda,x)$ in Ω by the strong maximum principle.

By Theorem 3.3 we know that Λ is an interval. The remainder of this section and the next will be devoted to estimating $\lambda^* = \sup \Lambda$.

Corollary 3.4. *Suppose that f satisfies F1–F4 and there exist a function* $h:\bar{\Omega} \to (\overline{\mathbf{R}^{+}})^k$ *and matrix* $M = ((m^{\mu\nu}(x)))$ *satisfying the hypotheses of part i) of Lemma 2.1 such that for each* u, v *with* $v \geq u \geq 0$, *we have for* $x \in \Omega$, $\mu = 1, \ldots, k$, *that*

$$f^{\mu}(x,u) + Ku^{\mu} \leq h^{\mu}(x) + \sum_{\nu=1}^{k} m^{\mu\nu}(x)v^{\nu} + Kv^{\mu}. \tag{3.6}$$

Then $\lambda^* \geq \lambda_0(M)$ *where* $\lambda_0(M)$ *is the first eigenvalue for (2.3).*

Proof. If $\lambda < \lambda_0(M)$ then by Lemma 2.1 the problem $Lu = \lambda(h(x) + M(x)u)$ in Ω, $u = 0$ on $\partial\Omega$, has a solution with each component nonnegative. Then Lemma 3.2 implies that (3.1) has a positive solution for that value of λ. Hence, $\lambda^* \geq \lambda_0(M)$.

Remark. If we can choose $M = 0$ then we may set $\lambda_0(0) = \infty$ and the result remains true.

Corollary 3.5. *Suppose that f satisfies F1–F4 and there exist a function* $h:\bar{\Omega} \to (\overline{\mathbf{R}^{+}})^k$, *with* $h^{\mu}(x) > 0$ *for some* μ, *some* $x \in \Omega$, *and a* $k \times k$ *matrix* $M = ((m^{\mu\nu}(x)))$ *satisfying the hypotheses of part (ii) of Lemma 2.1 such that for each* u, v *with* $v \geq u \geq 0$, *we have for* $x \in \Omega$, $\mu = 1, \ldots, k$ *that*

$$h^{\mu}(x) + \sum_{\nu=1}^{k} m^{\mu\nu}(x)u^{\nu} + Ku^{\mu} \leq f^{\mu}(x,v) + Kv^{\mu}. \tag{3.7}$$

Then $\lambda^* \leq \lambda_0(M)$.

Proof. The hypotheses of Lemma 2.1, part (ii), together with the fact that $h^{\mu}(x) > 0$ for some $x \in \Omega$ for some μ, imply that $F(x,v) = h(x) + M(x)v$ satisfies F1–F4. If $\lambda \in \Lambda$, it follows by Lemma 3.2 that the problem $Lu = \lambda h + \lambda Mu$ in Ω, $u = 0$ on $\partial\Omega$, has a nontrivial solution. But by Lemma 2.1, such a solution is only possible for $\lambda < \lambda_0(M)$, so $\lambda^* \leq \lambda_0(M)$.

In general, (3.1) may not have positive solutions unless $f^{\nu}(x,0) > 0$ for some ν and some $x \in \Omega$. This is noted in [4] for the single equation case. Another possibility is that condition F2 may not hold for all μ. In that case, our solution $\underline{u}(\lambda,x)$ might have $\underline{u}^{\mu}(\lambda,x) > 0$ in Ω only for those values of μ for which F2 is satisfied. In any case, we can weaken the conditions on the matrix occurring in Corollary 3.5 by using Corollary 2.3. The proof of the following result is the same as that of Corollary 3.5.

Corollary 3.6. *Suppose that f satisfies F1–F4, and* $M = ((m^{\mu\nu}(x)))$ *is a* $k \times k$ *matrix satisfying the hypotheses of Corollary 2.3. Let* d_i *be as in Corollary 2.3, and suppose that* $h:\bar{\Omega} \to (\overline{\mathbf{R}^{+}})^k$ *is such that for each* i, $h^{\mu}(x) > 0$ *for some* x *in* Ω *for some* $\mu = d_i + m$, $m \in \{1,\ldots,r_i\}$. *If for each* v *with* $v \geq u \geq 0$ *we have*

$$h^{\mu}(x) + \sum_{\nu=1}^{k} m^{\mu\nu}(x)u^{\nu} + Ku^{\mu} \leq f^{\mu}(x,v) + Kv^{\mu} \tag{3.8}$$

then $\lambda^* \leq \lambda_0(M^{\gamma})$ *for* $\gamma = 1, \ldots, N$.

Remark. The major advantage of Corollary 3.6 is that it is possible to apply it in situations where, for example, $f^{\mu}(x,u) = f^{\mu}(x,u^{\mu})$. Since Lemma 2.1 requires irreducibility on M, the sum $\sum_{\nu=1}^{k} m^{\mu\nu}u^{\nu}$ contains at least one nonzero term other than $m^{\mu\mu}u^{\mu}$, so if $f^{\mu}(x,v)$ depends only on v^{μ}, it is impossible to obtain inequality (3.7). However, a diagonal matrix can satisfy the hypotheses of Corollary 2.3, and could be used to obtain (3.8).

4. Concave and convex nonlinearities. In this section we shall require that f be C^2 in x and u. Let

$$f_u(x,u) = \left(\frac{\partial f^{\alpha}(x,u)}{\partial u^{\beta}}\right)_{\alpha,\beta=1}^{k}.$$

We then make the following assumptions:

F5 If $u \geq 0$, $\dfrac{\partial f^{\alpha}(x,u)}{\partial u^{\beta}} \geq 0$ for α, $\beta = 1, \ldots, k$, and for all $x \in \Omega$.

F6 If $u >> 0$, $f_u(x,u)$ is a nonnegative irreducible matrix with

$$\frac{\partial f^{\alpha}(x,u)}{\partial u^{\alpha}} > 0 \qquad \text{for } \alpha = 1, \ldots, k.$$

We now have

Theorem 4.1. *Let $f(x,u)$ satisfy F1, F2, F5 and F6, and let $\lambda^* = \sup \Lambda$, where Λ is as in Theorem 3.1. Then for each $\lambda \in (0,\lambda^*)$, $\lambda \leq \mu_1(\lambda)$ where $\mu_1(\lambda) = \mu_1\{f_u(x,\underline{u}(\lambda,x))\}$ is the principal eigenvalue of*

$$\begin{aligned} L\psi &= \mu f_u(x,\underline{u}(\lambda,x))\psi, & x &\in \Omega \\ \psi &\equiv 0, & x &\in \partial\Omega. \end{aligned} \tag{4.1}$$

Proof. First observe that if $\{\lambda_\nu\}_{\nu=1}^{\infty}$ is a monotone increasing sequence of positive numbers converging to $\lambda \in \Lambda$, $\underline{u}(\lambda_\nu,x)$ is a strictly increasing sequence of functions bounded above by $\underline{u}(\lambda,x)$ by Theorem 3.3. It follows as in the proof of Theorem 3.1 that $\underline{u}(\lambda_\nu,x)$ converges uniformly to a componentwise positive function $\underline{\underline{u}}(\lambda,x)$, which solves (3.1), and is such that $\underline{\underline{u}}(\lambda,x) \leq \underline{u}(\lambda,x)$. Hence $\underline{\underline{u}}(\lambda,x) = \underline{u}(\lambda,x)$ and $\underline{u}$ is left-continuous in λ on Λ.

Now let $0 < \lambda' < \lambda \in \Lambda$. Define, as in [8],

$$f_u(\lambda,\lambda',x) = \int_0^1 f_u(x,\theta\underline{u}(\lambda,x) + (1-\theta)\underline{u}(\lambda',x))d\theta.$$

Note that

$$\frac{d}{d\theta} f(x,\theta\underline{u}(\lambda,x) + (1-\theta)\underline{u}(\lambda',x)) = [f_u(x,\theta\underline{u}(\lambda,x) + (1-\theta)\underline{u}(\lambda',x))]\cdot[\underline{u}(\lambda,x) - \underline{u}(\lambda',x)].$$

So

$$f(x,\underline{u}(\lambda,x)) - f(x,\underline{u}(\lambda',x))$$
$$= \left(\int_0^1 [f_u(x,\theta\underline{u}(\lambda,x) + (1-\theta)\underline{u}(\lambda',x))]d\theta\right)\cdot(\underline{u}(\lambda,x) - \underline{u}(\lambda',x))$$
$$= f_u(\lambda,\lambda',x)\cdot(\underline{u}(\lambda,x) - \underline{u}(\lambda',x)).$$

Hence

$$L[\underline{u}(\lambda,x) - \underline{u}(\lambda',x)] = \lambda f(x,\underline{u}(\lambda,x)) - \lambda' f(x,\underline{u}(\lambda',x))$$
$$= \lambda f_u(\lambda,\lambda',x)\cdot(\underline{u}(\lambda,x) - \underline{u}(\lambda',x)) + (\lambda - \lambda')f(x,\underline{u}(\lambda',x)).$$

Since f satisfies F5–F6, $f_u(\lambda,\lambda',\cdot)$ satisfies the hypotheses of Lemma 2.1. Hence $\lambda < \mu_1\{f_u(\lambda,\lambda',x)\}$ since $\underline{u}(\lambda,x) - \underline{u}(\lambda',x) >> 0$ on Ω. Since $\underline{u}(\lambda',x)$ converges to $\underline{u}(\lambda,x)$ uniformly on $\bar{\Omega}$ as $\lambda' \uparrow \lambda$, it follows that

$$f_u(\lambda,\lambda',\cdot) \to f_u(\cdot,\underline{u}(\lambda,\cdot)) \qquad \text{(strongly)}$$

as operators on $[C^\alpha(\bar{\Omega})]^k$. (The details of the proof of this result are as in the proof of Theorem 4.4). Thus $\{L^{-1}f_u(\lambda,\lambda',\cdot)\}_{\lambda'\uparrow\lambda}$ forms a left-continuous (in the strong operator topology) family of compact positive operators on $[C^\alpha(\Omega)]^k$ with

$$L^{-1}f_u(\lambda,\lambda',\cdot) \to L^{-1}f_u(\cdot,\underline{u}(\lambda,\cdot)) \qquad \text{as } \lambda' \uparrow \lambda.$$

It follows then from a result of Nussbaum [11] that

$$\mu_1\{f_u(\lambda,\lambda',x)\} \to \mu_1\{f_u(x,\underline{u}(\lambda,x))\} \qquad \text{as } \lambda' \uparrow \lambda.$$

Thus $\lambda \le \mu_1\{f_u(x,\underline{u}(\lambda,x))\}$.

In order to examine the function $\mu_1 : \Lambda \to \mathbf{R}$ more closely, we need further hypotheses on f. To this end, we shall say that $f(x,u)$ is *concave* if f satisfies F5 and F6 and

F7
$$\frac{\partial f^\alpha}{\partial u^\beta}(x,w) \le \frac{\partial f^\alpha}{\partial u^\beta}(x,w')$$

for $w >> w' \ge 0$ if α, $\beta = 1, \ldots, k$, with strict inequality if $\alpha = \beta$.

F8 $\left(\dfrac{\partial^2 f^\alpha}{\partial u^\beta \partial u^\gamma}(x,w)\right)_{\beta,\gamma=1}^k$ is negative semidefinite for $x \in \bar{\Omega}$ and $w \ge 0$, for each $\alpha = 1, 2, \ldots, k$.

Remark. Concavity is the geometric requirement that f^α lie below any of its tangent hypersurfaces for each $\alpha = 1, \ldots, k$. This requirement can be expressed analytically without the assumption that f is C^2. However, such an expression is complicated, and confers little advantage in view of the fact that f must be $C^{1+\alpha}$ in (x,u) in any case. One should also observe that in the single equation case [8], F7 alone reflects concavity. However, F7 and F8 coincide when f is C^2 in this situation.

Analogously, we say that f is *convex* if f satisfies F5 and F6 and

F9 $$\frac{\partial f^\alpha}{\partial u^\beta}(x,w) \geq \frac{\partial f^\alpha}{\partial u^\beta}(x,w')$$

for $w >> w' \geq 0$ if $\alpha, \beta = 1, \ldots, k$, with strict inequality if $\alpha = \beta$.

F10 $\left(\frac{\partial^2 f^\alpha}{\partial u^\beta \partial u^\gamma}(x,w)\right)_{\beta,\gamma=1}^{k}$ is positive semidefinite for $x \in \bar{\Omega}$ and $w \geq 0$, for each $\alpha = 1, 2, \ldots, k$.

Notice that if $u >> 0, f^\alpha(x,u) = f^\alpha(x,0) + f^\alpha_u(x,u^*_\alpha)u$, where $0 << u^*_\alpha << u$. If f is concave, it follows that

$$f(x,u) << f(x,0) + f_u(x,0)u, \qquad \text{if } u >> 0,$$

and if f is convex, that

$$f(x,u) >> f(x,0) + f_u(x,0)u \qquad \text{for } u >> 0.$$

In the concave case, if f also satisfies F1 and F2, Corollary 3.4 implies $(0,\mu_1(f_u(x,0))) \subseteq \Lambda$ and $\mu_1(0) = \mu_1(f_u(x,0)) \leq \lambda^*$. In case f is convex, F1 and F2 hold, and also $f_u(x,0)$ satisfies (ii) of Lemma 2.1, $\lambda^* \leq \mu_1(0)$ by Corollary 3.5. (Note that in the concave situation $f_u(x,0)$ automatically satisfies (ii) of Lemma 2.1. In the sequel, this extra condition will be tacitly assumed in the convex case.) Sharper estimates are given in

Corollary 4.2. *Suppose that $f(x,u)$ satisfies F1, F2, F5, F6, (F7, F8) or (F9, F10). Then if $\Lambda = (0,\lambda^*)$, $\mu_1(\lambda)$ is an (increasing) (or decreasing) function of λ on this interval. Furthermore, $\mu_1(\lambda) < \lambda^*$ if f is concave and $\mu_1(\lambda) > \lambda^*$ for f convex.*

Proof. Suppose $f(x,u)$ is concave and let $0 < \lambda < \lambda' < \lambda^*$. Theorem 3.3 implies that $\underline{u}(\lambda,x) << \underline{u}(\lambda',x)$. Let $\psi >> 0$ be a solution of

$$L\psi = \mu_1(\lambda) f_u(x,\underline{u}(\lambda,x))\psi \qquad \text{in } \Omega$$
$$\psi = 0 \qquad \text{on } \partial\Omega.$$

Then

$$L\psi = \mu_1(\lambda) f_u(x,\underline{u}(\lambda',x))\psi + \mu_1(\lambda)[f_u(x,\underline{u}(\lambda,x)) - f_u(x,\underline{u}(\lambda',x)]\psi.$$

Hypothesis F7 implies $[f_u(x,\underline{u}(\lambda,x)) - f_u(x,\underline{u}(\lambda',x))]\psi$ is nonnegative and nontrivial. Lemma 2.1(ii) then implies that $\mu_1(\lambda) < \mu_1(\lambda')$. An analogous argument will give $\mu_1(\lambda) > \mu_1(\lambda')$ in case F9 holds.

Suppose f is convex and $\mu_1(\lambda_0) \leq \lambda^*$ for some $\lambda_0 \in (0,\lambda^*)$. Then if $\lambda \in (\lambda_0,\lambda^*)$, $\mu_1(\lambda) < \lambda^*$. Let $\lambda \in (\lambda_0,\lambda^*)$ and choose $\lambda' \in (\mu_1(\lambda), \lambda^*)$. Then $\lambda \leq \mu_1(\lambda) < \lambda'$ implies $\mu_1(\lambda') < \mu(\lambda) < \lambda'$, a contradiction since $\lambda' \leq \mu_1(\lambda')$. Hence $\mu_1(\lambda) > \lambda^*$ for all $\lambda \in (0,\lambda^*)$.

Suppose f is concave and $\phi(x) >> 0$. Then

$$(4.2)\quad f^\alpha(x,\phi(x)) = f^\alpha(x,\underline{u}(\lambda,x)) + \sum_{\beta=1}^{k} \frac{\partial f^\alpha}{\partial u^\beta}(x,\underline{u}(\lambda,x)) \cdot [\phi^\beta(x) - \underline{u}^\beta(\lambda,x)]$$
$$+ \frac{1}{2} \sum_{\beta,\gamma=1}^{k} \frac{\partial^2 f^\alpha}{\partial u^\beta \partial u^\gamma}(x,w^\alpha(x)) \cdot [\phi^\beta(x) - \underline{u}^\beta(\lambda,x)] \cdot [\phi^\gamma(x) - \underline{u}^\gamma(\lambda,x)]$$

where $w^\alpha \geq 0$, $\alpha = 1, 2, \ldots, k$. F8 implies that

$$f^\alpha(x,\phi(x)) \leq f^\alpha(x,\underline{u}(\lambda,x)) + \sum_{\beta=1}^{k} \frac{\partial f^\alpha}{\partial u^\beta}(x,\underline{u}(\lambda,x)) \cdot [\phi^\beta(x) - \underline{u}^\beta(\lambda,x)]$$

and hence $f(x,\phi) \leq f(x,\underline{u}) + f_u(x,\underline{u})(\phi - \underline{u})$. Then F7 yields

$$(4.3)\quad f(x,\phi) \leq f(x,0) + [f_u(x,0) - f_u(x,\underline{u})]\underline{u} + f_u(x,\underline{u})\phi.$$

It then follows that $\mu_1(\lambda) \leq \lambda^*$ for $\lambda \in (0,\lambda^*)$. Since $\mu_1(\lambda)$ is strictly increasing, it is then necessarily the case that $\mu_1(\lambda) < \lambda^*$ on Λ.

We next consider the behavior of $u(\lambda,x)$ and $\mu_1(\lambda)$ as $\lambda \uparrow \lambda^*$ in the case of concave nonlinearities. The following lemma is an extension to systems of part of Theorem 1.1 of [5].

Lemma 4.3. *Suppose that $f(x,u)$ satisfies F1, F2, F5 and F6, that $\lambda^* < \infty$, and there exists a constant M such that $\sup|\underline{u}(\lambda,x)| < M$ for $\lambda \in \Lambda$. Then $\lambda^* \in \Lambda$ and $\underline{u}(\lambda,x) \to \underline{u}(\lambda^*,x)$ as $\lambda \uparrow \lambda^*$.*

Proof. The proof follows that of Theorem 1.1 of [5]; it is also related to the proofs of Theorems 3.1 and 4.1 of the present article; so we will only sketch the argument. Suppose $\sup_\Omega|\underline{u}(\lambda,x)| < M$; then $\|\underline{u}^\mu(\lambda,x)\|_{L^p(\Omega)}$ is uniformly bounded in λ for any p, μ, so that $f(x,\underline{u}(\lambda,x))$ is uniformly bounded in $(L^p(\Omega))^k$ for any p. Standard elliptic a priori estimates then imply that $\underline{u}(\lambda,x))$ is uniformly bounded in $(W^{2,p}(\Omega))^k$ for any p, and hence by the Sobolev embedding theorem in $(C^{1+\alpha}(\bar{\Omega}))^k$ for any $\alpha \in (0,1)$. Hence, $f(x,\underline{u}(\lambda,x))$ is uniformly bounded in $(C^\alpha(\bar{\Omega}))^k$, and thus $\underline{u}(\lambda,x)$ is uniformly bounded in $(C_0^{2+\alpha}(\bar{\Omega}))^k$. If we choose a sequence $\lambda_n \uparrow \lambda^*$, we observe that $\{\underline{u}(\lambda_n,x)\}$ is a componentwise increasing sequence that has a subsequence convergent in $(C^\alpha(\bar{\Omega}))^k$ and hence, via the differential equation and standard elliptic estimates, in $(C_0^{2+\alpha}(\bar{\Omega}))^k$. It follows that the limiting function of the subsequence satisfies (3.1) with $\lambda = \lambda^*$.

Remark. In [5], the result corresponding to the above lemma is used to study convex nonlinearities in the case of a single equation. In such a case, it is sometimes possible to give conditions on the nonlinearity which imply a priori bounds on the solution; see for example [1], [5]. Similar arguments could be made for systems in some cases, but we shall not attempt to do so here.

In the case of a single equation, the solution set for convex nonlinearities can be much more complicated than that for concave nonlinearities. Specifically, the solution in the concave case turns out to be unique, but for some convex nonlinearities the existence of multiple solutions can be shown. Results in that direction for a single equation are given in [5], [7], [10]. In the present article we do not consider the behavior of $\mu_1(\lambda)$ and $\underline{u}(\lambda,x)$ as $\lambda \uparrow \lambda^*$ for the convex case, nor the question of multiple solutions in that case. We intend to study such questions in later work. We have the following results in the concave case:

Theorem 4.4. *Suppose that $f(x,u)$ satisfies F1, F2, F5, F6, F7, and F8. Then* $\lim_{\lambda \uparrow \lambda^*} \mu_1(\lambda) = \lambda^*$, $\lambda^* \notin \Lambda$, *and if* $\lambda^* < \infty$, $\sup_\Omega |\underline{u}(\lambda,x)| \to \infty$ *as* $\lambda \uparrow \lambda^*$.

Proof. As shown in the proof of Theorem 4.1, $\underline{u}(\lambda,x)$ is continuous from the left in λ in $(C(\bar{\Omega}))^k$, hence (via elliptic a priori estimates as in the proof of Lemma 4.3) in $(C^{1+\alpha}(\bar{\Omega}))^k$, so that the operators $L^{-1}f_u(\cdot,\underline{u}(\lambda,\cdot))$ form a left-continuous family of compact positive operators on $(C^\alpha(\bar{\Omega}))^k$. It follows by a result of Nussbaum [11] that $\mu_1(\lambda)$ is left-continuous. From Corollary 4.2, we have $\mu_1(\lambda) < \lambda^*$ and from Theorem 4.1 we have $\lambda \le \mu_1(\lambda)$, so by the left-continuity of $\mu_1(\lambda)$, we have $\lim_{\lambda \uparrow \lambda^*} \mu_1(\lambda) = \lambda^*$.

Suppose that $\lambda^* \in \Lambda$; then consider $\underline{u}(\lambda^*,x)$. Since $\mu_1(\lambda)$ and $\underline{u}(\lambda,x)$ depend continuously on the left on λ, we have $\lambda^* = \mu_1\{f_u(x,\underline{u}(\lambda^*,x))\}$. Also, by the monotonicity of $\underline{u}(\lambda,x)$, we have $\underline{u}(\lambda,x) \le \underline{u}(\lambda^*,x)$ for all $\lambda \in \Lambda$. If ϕ and ψ are vectors in $(\overline{\mathbf{R}^+})^k$, $\psi >> 0$, then arguing as in (4.2), (4.3) leads to the inequality

$$f(x,\phi) \le f(x,0) + [f_u(x,0) - f_u(x,\psi)]\psi + f_u(x,\psi)\phi \tag{4.4}$$

with $f(x,0) + [f_u(x,0) - f_u(x,\psi)]\psi >> 0$. If we choose $\psi \in (C^{2+\alpha}(\bar{\Omega}))^k$ with $\psi >> \underline{u}(\lambda^*,x)$, then arguing as in the proof of Corollary 4.2 yields

$$\mu_1\{f_u(x,\psi)\} > \mu_1\{f_u(x,\underline{u}(\lambda^*,x))\} = \lambda^*.$$

But (4.4) and Corollary 3.4 imply that $\mu_1\{f_u(x,\psi)\} \le \lambda^*$; this is a contradiction, so $\lambda^* \notin \Lambda$. Finally, suppose $\lambda^* < \infty$; if

$$\sup_{\lambda \in \Lambda} \sup_{x \in \bar{\Omega}} |\underline{u}(\lambda,x)| = M < \infty,$$

then by Lemma 4.3, $\lambda^* \in \Lambda$. By the above argument, $\lambda^* \notin \Lambda$, so we must have $\sup_{\lambda \in \Lambda} \sup_{x \in \bar{\Omega}} |\underline{u}(\lambda,x)| = \infty$. Since $\lambda' > \lambda$ implies $\underline{u}(\lambda',x) >> \underline{u}(\lambda,x)$, it follows $\sup_{x \in \bar{\Omega}} |\underline{u}(\lambda,x)| \to \infty$ as $\lambda \uparrow \lambda^*$.

We now consider the question of uniqueness for concave and certain other nonlinearities. As noted previously, the question of uniqueness in the convex case is much more complicated and we will not pursue it here.

Theorem 4.5. *Suppose $f(x,u)$ satisfies F1, F2, F5, F6, and F7. Then the minimal solution $\underline{u}(\lambda,x)$ is the only positive solution of (3.1) for $\lambda \in \Lambda$.*

Proof. Fix $\lambda \in \Lambda$; if u is any solution to (3.1) for that value of λ, $u \ge \underline{u}$ by Theorem 3.1. We have by F7 and (3.1) that

$$L(u - \underline{u}) = \lambda(f(x,u) - f(x,\underline{u})) \tag{4.5}$$
$$= \lambda\left[\int_0^1 f_u(x,\theta u + (1-\theta)\underline{u})d\theta\right]\cdot(u - \underline{u})$$
$$= \lambda f_u(x,\underline{u})(u - \underline{u}) + g(x)$$

where

$$g(x) = \lambda\left[\int_0^1 f_u(x,\theta u + (1-\theta)\underline{u})d\theta \cdot - f_u(x,\underline{u})\right](u - \underline{u}).$$

We have $f_u(x,w)$ nonnegative irreducible for $w \geq 0$, so that the integral in the second line of (4.5) is a nonnegative irreducible matrix. It follows by the strong maximum principle that $u >> \underline{u}$. Thus we have

$$\theta u + (1-\theta)\underline{u} >> \theta\underline{u} + (1-\theta)\underline{u} = \underline{u}$$

for all $\theta \in (0,1]$, so

$$\frac{\partial f^\alpha}{\partial u^\beta}(x,\theta u + (1-\theta)\underline{u}) \leq \frac{\partial f^\alpha}{\partial u^\beta}(x,\underline{u}).$$

Hence

$$\left[\int_0^1 f_u(x,\theta u + (1-\theta)u)d\theta\right](u - \underline{u}) \leq \left[\int_0^1 f_u(x,\underline{u})d\theta\right](u - \underline{u}) = f_u(x,\underline{u})(u - \underline{u}),$$

so that $g(x) \leq 0$.

By Theorem 4.1, $\lambda \leq \mu_1(\lambda)$; if $\lambda < \mu_1(\lambda)$, then since $u - \underline{u} = 0$ on $\partial\Omega$, it follows by Lemma 2.1 that $u - \underline{u} \leq 0$. However, $u - \underline{u} \geq 0$ since $\underline{u}$ is the minimal solution, so $u = \underline{u}$. If $\lambda = \mu_1(\lambda)$ it follows as in the proof of Lemma (2.1) that (4.5) can have no componentwise nonnegative solution with $g(x) \leq 0$ unless $g(x) \equiv 0$; but then F7 implies $u \equiv \underline{u}$.

The following result implies uniqueness in some cases where the nonlinearity need not be concave:

Theorem 4.6. *Suppose $f(x,u)$ satisfies F1, F2, F5, F6, and for some matrix $M(x)$ satisfying the hypotheses of (i) of Lemma 2.1, we have*

$$\frac{\partial f^\mu(x,u)}{\partial u^\nu} \leq M^{\mu\nu}(x)$$

for $u \in (\overline{\mathbf{R}^+})^k$, $x \in \bar{\Omega}$. Then the minimal solution $\underline{u}(\lambda,x)$ is the only solution of (3.1) for $\lambda \in (0,\lambda_0(M))$.

Proof. Arguing as in (4.5), we have for any fixed $\lambda \in (0,\lambda_0(M))$ and any solution u other than $\underline{u}$ that $u - \underline{u} \geq 0$, and

$$L(u - \underline{u}) = \lambda[f(x,u) - f(x,\underline{u})] = \lambda M(x)(u - \underline{u}) + g(x)$$

with

$$g(x) = \lambda\left[\int_0^1 f_u(x,\theta u + (1-\theta)\underline{u})d\theta - M(x)\right](u - \underline{u}) \leq 0.$$

Hence, since $u - \underline{u} = 0$ on $\partial\Omega$, it follows by Lemma 2.1 that $u - \underline{u} \leq 0$ so that $u = \underline{u}$ as desired.

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